PMM U.S.S.R.,Vol.52,No.5,pp.677-679,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

SPATIALLY MODULATED CONVECTIVE MOTIONS IN A VERTICAL LAYER WITH CURVED BOUNDARIES*

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Mixed convection modes in a vertical layer with periodically curved boundaries are investigated. The amplitude of the waviness of the layer walls and the flow of fluid along the layer are both assumed to be small, and the value of the Grashof number is assumed to be close to critical. It is established that, in addition to the spatially periodic motions where the amplitude of the waviness is small, studied in /1, 2/, stable spatially modulated (quasiperiodic and soliton-type) wave modes of flow are possible, resembling the modulated convective structures in a horizontal layer /3/.

1. The problem was formulated earlier in /1, 2/. Confining ourselves below to the region $G > G_c$ ($G = g\beta\Theta d^3/\nu^2$ is the Grashof number and G_c is the critical Grashof number), we use the following transformations:

$$T = \frac{JG_2}{I} t_2, \quad Y = \left(\frac{JG_2}{R}\right)^{1/2} y_1 \tag{1.1}$$
$$A = \left(\frac{S}{JG_2}\right)^{1/2} a_1 \exp(-ik_1y_1)$$

to reduce Eq.(2.9) of /2/ describing the evolution of the envelope system of convective vortices, to the following form (a subscript denotes differentiation with respect to the corresponding variable):

$$A_T = A_{YY} + 2iK_0A_Y + (1 - K_0^2 - i\omega) A - |A|^2 A + \delta \tag{1.2}$$
$$Y \to \pm\infty, \quad |A| < \infty$$

Here

$$\omega = -\frac{Bq}{J(G - G_c)}, \quad \delta = \frac{DS^{1/2}}{J^{3/2}} \frac{\eta}{(G - G_c)^{3/2}} \tag{1.3}$$
$$K_0 = \left(\frac{R}{J}\right)^{1/2} \frac{k_0 - k_c}{(G - G_c)^{1/2}}$$

(q is the dimensionless flow of fluid, k_0, η is the wave number and amplitude of the waviness of the boundaries, k_c is the critical wave number, and B, J, D, S, R are real constants defined in /2, 4/).

A particular class of solutions of Eq.(1.2) of the form $A = Z(T)$ was studied in detail in /1, 2/. This type of solution corresponds, in accordance with the definition of the amplitude function, to spatially periodic motions with period $2\pi/k_0$, "imposed" by the waviness of the boundaries. Below we shall discuss other types of stable motions without spatial periodicity for small values of the parameter δ.

2. We shall seek solutions of problem (1.2) periodic in Y and T, in the form of a series in powers of δ:

$$A = \sum_{n=0}^{\infty} \delta^n \sum_{m=-n+1}^{n+1} A_m^{(n)} \exp[im(\Delta Y - \Omega T + \varphi_0)] \tag{2.1}$$
$$\Omega = \sum_{n=0}^{\infty} \delta^n \Omega^{(n)}$$

choosing, as the zeroth approximation,

$$A_1^{(0)} = (1 - K^2)^{1/2}; \quad \Omega^{(0)} = \omega; \quad K = K_0 + \Delta$$

Let us substitute (2.1) into (1.2) and equate terms of like powers in δ. It can be shown

*Prikl.Matem.Mekhan.52,5,863-867,1988

that when the condition $K^2 < 1/3$ holds, the problem has a solution in all powers of δ except in the resonant case $\Delta = \omega = 0$. The quantities $A_m^{(n)}$ are non-zero only if m and n are of different parity, and the expansion for Ω contains only terms with even n. Note that the coefficients $A_m^{(n)}$ and $\Omega^{(n)}$ contain the factor $|H^{-n}|$, where

$$H = 2(1 - 3K^2)\Delta^2 + 2i\omega(1 - K^2) + (\Delta + i\omega)^2$$

and therefore the small parameter over which the functions are expanded, is in fact $\delta|H|^{-1}$.

Solution (2.1) describes a travelling wave with mean wave number $k_c + KR^{-1/2}[(J(G - G_c)]^{1/2}$, spatially modulated with period $2\pi R^{1/2}[J\Delta(G - G_c)]^{-1/2}$.

In order to investigate the stability, we shall impose on periodic solution A a small perturbation a whose evolution is described by the problem with periodic coefficients. The normal perturbations have the form of the Floquet functions

$$a = a_1(Y, T)\exp(iQY + \lambda T) + a_2(Y, T)\exp(-iQY + \lambda T)$$
$$a_i(Y + 2\pi\Delta^{-1}, T) = a_i(Y, T)$$
$$a_i(Y, T + 2\pi\Omega^{-1}) = a_i(Y, T); \quad i = 1, 2$$

The most dangerous perturbations are those with small Q which transform, as $Q \to 0$, into a neutral perturbation $a_1 = a_2 = A_Y$, $\lambda = 0$ corresponding to an infinitesimal change in the constant φ_0. For these perturbations

$$\mathrm{Re}\lambda = sQ^2 + \ldots, \quad s = -(1 - 3K^2)(1 - K^2)^{-1} + \delta^2 s^{(2)} + \ldots \tag{2.2}$$

(the explicit expression for $s^{(2)}(K, K_0, \omega)$ is bulky and will not be given here). From expression (2.2) it follows that the solutions (2.1) are stable for small δ when $K_- < K < K_+$, where

$$K_\pm = \pm\frac{1}{\sqrt{3}}\left[1 - \frac{1}{3}\delta^2 s^{(2)}\left(\pm\frac{1}{\sqrt{3}}, K_0, \omega\right)\right]$$

3. As we said before, the ratio $\delta/|H|$ serves as the small parameter in the expansions (2.1). In the resonant case $|H| \sim \delta$, i.e. $\Delta \sim \delta^{1/2}$, $\omega \sim \delta$), the approach described in Sect.2 becomes ineffective. In this case, however, we can use the fact that the amplitude of A is a slow function of the spatial coordinate and time.

Confining ourselves to the case $K_0^2 < 1/3$ we obtain, as in /2/, the evolutionary equation for the phase Φ of the complex variable A:

$$\Phi_\tau = \Phi_{XX} - \sin\Phi - F; \quad X \to \pm\infty, \quad |\Phi_X| < \infty \tag{3.1}$$
$$\tau = \delta T(1 - K_0^2)^{-1/2}, \quad F = \delta^{-1}\omega(1 - K_0^2)^{1/2}$$
$$X = \delta^{1/2}Y(1 - K_0^2)^{1/4}(1 - 3K_0^2)^{-1/2}$$

The solutions of the problem independent of the coordinate X correspond to the spatially periodic motions studied earlier in /1, 2/. Here we shall discuss new classes of flows described by solutions which depend on the X coordinate.

Let us first consider the spatially modulated stationary flows. When $\Phi_\tau = 0$, Eq.(3.1) will describe the motion of a pendulum acted upon by a constant moment of force (the X coordinate will serve as time), and its solutions will be written in terms of elliptic functions.

In order to determine the stability of the modulated stationary motions, we must investigate the spectrum of the linearized problem for the perturbations $\varphi(X)\exp\lambda\tau$ imposed on the solution Φ, which represents the Sturm-Liouville problem (the Schrödinger equation). It is well-known that the largest value of λ corresponds to the function φ of constant sign. At the same time, the equation will always have the solution $\varphi = \Phi_X, \lambda = 0$. This implies that the stationary motion is stable if the function $\Phi(X)$ is monotonic. When $F = 0$, the soliton solutions

$$\Phi = 4\,\mathrm{arctg}\, e^{\pm X} \tag{3.2}$$

are stable together with the solutions with $\Phi(X + L) = \Phi(X) \pm 2\pi$, and solutions for which $\Phi(X + L) = \Phi(X)$, are unstable. When $F \neq 0$, all stationary solutions except $\Phi = \Phi_+ = -\arcsin F$ ($|F| < 1$) are unstable.

Let us turn our attention to the solutions

$$\Phi = \Phi(\xi), \quad \xi = X - c\tau \tag{3.3}$$

which correspond to modulated travelling waves. We will write problem (3.1) in the form

$$\Phi_{\xi\xi} + c\Phi_\xi - \sin\Phi - F = 0; \quad \xi \to +\infty, \quad |\Phi_\xi| < \infty \tag{3.4}$$

Eq.(3.4) describes the motion of a pendulum with friction, under the action of a constant moment of force /5/. A similar problem was studied in the course of investigating the properties of the soliton lattices in distributed Josephson junctions /6, 7/. The domains of existence of the various solutions are shown in the figure; the coordinates of the point M: $F = 1$, $c = 1.19$ /7/. Periodic solutions of problem (3.4) exist in regions 1 and 2; for a given value of F the period increases monotonically as c increases. When $F > 1$, the periodic solutions (3.3) become, as $c \to \infty$,

$$\Phi = -2\mathrm{Arctg}\,\{(1 + (F^2 - 1)^{1/2} \cdot \mathrm{tg}\,[2(F^2 - 1)^{1/2}(\tau - \tau_0)])F^{-1}\}$$

independent of the X coordinate. If on the other hand $F < 1$ and the spatial period increases without limit, the phase velocity tends to some limit value. A soliton solution exists at the boundary separating regions 1 and 3, and it tends as $\xi \to \pm\infty$ to the limit values

$$\Phi(-\infty) = \Phi_+ - 2\pi, \quad \Phi(+\infty) = \Phi_+ \tag{3.5}$$

We note that near this boundary, the solutions with a finite but large period become similar to a periodic lattice of solitons. A soliton also exists, with the limit value of (3.5), at the boundary between regions 2 and 3, but its asymptotic form at infinity is of a power type. Finally, a soliton solution exists in regions 1 and 3 with the asymptotic form

$$\Phi(-\infty) = \Phi_+, \quad \Phi(+\infty) = \Phi_- = \arcsin F - \pi \tag{3.6}$$

and in region 3 we also have solitons with the asymptotic form

$$\Phi(-\infty) = \Phi_+ - 2\pi, \quad \Phi(+\infty) = \Phi_- \tag{3.7}$$

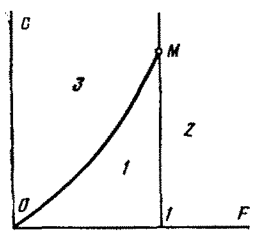
The above types of solutions describe the "front" of the displacement of the unstable motion with $\Phi = \Phi_-$ by the stable motion with $\Phi = \Phi_+$.

The stability of solutions (3.3) in a reference system moving with the wave, is determined by the spectrum of the boundary value problem

$$\lambda\varphi = \varphi_{\xi\xi} + c\varphi_\xi - \cos\Phi(\xi)\,\varphi, \quad \xi \to \pm\infty, \quad |\varphi_\xi| < \infty \tag{3.8}$$

When Φ are periodic, the functions φ have the form of Floquet functions. It is easy to establish the instability of the soliton solutions (3.6) and (3.7) related to the continuous spectrum. The stability of periodic solutions and solitons of the type (3.5) was established in /7/.

Thus we find that in a layer with wavy boundaries and low-amplitude waviness, we can have, in addition to spatially periodic motions with a wavelength related to the curvature of the boundaries, quasiperiodic stable motions (modulated waves) whose spatial spectrum contains a discrete set of wave numbers. When the outer parameters of the problem are fixed, the motions of the given class constitute a one-parameter family. If $|F|>1$ (the flow of fluid along the layer $|q|$ exceeds some critical value), then increasing the modulation period transforms them into the travelling, spatially periodic waves studied earlier in /1, 2/. If on the other hand $|F|<1$ (the flow of fluid is less than critical), then increasing the modulation period transforms the quasiperiodic solutions into a soliton solution with a continuous spatial Fourier spectrum. The solution approaches the periodic solution asymptotically at large distances, but contains a stable moving local defect at finite values of the spatial coordinate.



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Translated by L.K.